http://dx.doi.org/10.21611/qirt.2014.093

Inverse problem in determining the thermal diffusivity of materials by means of pulsed IR thermogaphy

by W. Oliferuk*, W. Kosiński**, K. Kochanowski* and A. Adamowicz*

* Białystok Technical University, Department of Mechanics and Applied Computer Science , Wiejska 45C, 15-351 Białystok, Poland, *w.oliferuk@pb.edu.pl*
** Polish-Japanese Institute of Information Technology, Computer Science Department, Koszykowa 86, 02-008 Warsaw, Poland, *wkos@pjwstk.edu.pl*

Abstract

The paper concerns two methods of inverse problem solution for the equation of heat conduction, what allows to determine the thermal diffusivity of the materials using pulsed infrared thermography. Both methods are related to finding the time dependence of the temperature of an infinite plate surface, when opposite surface of the plate was heated by a short heat pulse. This dependence is compared with the time evolution of the temperature of the rear plate surface measured by the means of an infrared (IR) camera. Such comparison allows to extract, from experimental data, the information about thermal diffusivity of the tested material.

1. Introduction

Progress in determining thermal properties of materials, such as thermal diffusivity, is possible owing to the development of the active infrared thermography techniques. Thermal diffusivity α characterizes the material in a complex way, because it includes the heat conductivity λ, the specific heat c_s and the mass density ρ of the material:

$$\alpha = \frac{\lambda}{c_s \rho}, \tag{1}$$

One of active thermography techniques is the pulsed infrared thermography (PIRT), based on stimulation of the plate surface by a short heat pulse (few milliseconds) and recording the material response, as a time evolution of the temperature distribution on the rear surface, by the means of an infrared (IR) camera. Such evolution contains information about the thermal diffusivity of the tested material. To obtain information about thermal diffusivity of the tested material one needs to use the inverse problem solution. In order to do this, the equation of heat conduction should be solved for the initial-boundary conditions consistent with the experiment.

Two methods: a) and b) of the equation solution are presented. The solution obtained by method a) has been compared with the experimental time evolution temperature of the rear surface measured by means of IR camera and thermal diffusivity of selected materials has been determined. Application of the method b) to determine the thermal diffusivity is still developed.

2. Method a)

2.1. Theoretical foundations of the method

The theoretical basis of determining thermal diffusivity of materials based on solutions of heat conduction equation formulated for a plate of a finite thickness when one of its surfaces is uniformly heated by a short heat pulse. If the surface of the plate is sufficiently large in comparison with the region of interest, it may be considered as infinite and then the one-dimensional model of heat conduction can be presumed (figure 1). It is assumed that the transport of heat by convection and radiation compared with the heat conduction mechanism are negligible.

The time evolution of temperature of the opposite surface with respect to the stimulated one is described by the solution of the heat conduction equation for the plate where $z = g$. This solution includes the thermal diffusivity of the material of the plate. Therefore, if the temperature of this surface is measured in time, it is possible to determine the thermal diffusivity of the tested material.

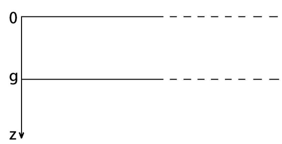

Fig. 1. *Schematic representation of the infinite homogeneous layer of material*

http://dx.doi.org/10.21611/qirt.2014.093

The differential equation of heat conduction for the one-dimensional model has the following form:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} + \frac{1}{\rho c_s} q, \tag{2}$$

Eq. (2) has been solved for relatively simple, homogeneous initial and boundary conditions:

$$T(t=0) = T_o, \quad \frac{\partial T}{\partial z}(z=0) = 0 \text{ and } \frac{\partial T}{\partial z}(z=g) = 0 \tag{3}$$

It is assumed than a heat pulse stimulation has the fallowing form:

$$q = \delta(t)\delta(z) \tag{4}$$

Using of the Fourier transform, the solution of the heat conduction equation for the plate, whose surface is heated by heat pulse has been obtained [1]. The dimensionless the solution, for z=1 has the form:

$$\vartheta^{(1)}\left(\bar{t}\right) = 1 + 2\sum_{k=1}^{\infty}(-1)^k \exp\left[(-k\pi)^2 \bar{t}\right] \tag{5}$$

The elements of the series are getting smaller with t, So, it can be seen that for a sufficiently large value of t, error omission of elements $k > 1$ does not exceed the error of temperature measurement. Thus, series can be reduced to the first element:

$$\vartheta^{(1)}\left(\bar{t}\right) = 1 + 2\exp\left[(-\pi)^2 \bar{t}\right]. \tag{6}$$

The dimensional form of the solution is as follows:

$$T(t) = T_{\max} - 2\left(T_{\max} - T_0\right) exp\left(-\frac{\pi^2 \alpha}{g^2} t\right), \tag{7}$$

T_{max} is the temperature of the surface after a long time since heat stimulation. The logarithm of Eq. 7 has a linear character with respect to time.

$$\ln(T_{\max} - T) = -\frac{\pi^2}{g^2} \cdot \alpha \cdot t + \ln 2(T_{\max} - T_0). \tag{8}$$

where

$$\frac{\pi^2 \alpha}{g^2} = A \tag{9}$$

is the tangent of the slope straight line to the timeline.
So, if one of the specimen surfaces will be heated by a short impulse of heat and the temperature of the opposite surface will be measured as a function of time, we can determine the thermal diffusivity α:

$$\alpha = \frac{g^2 A}{\pi^2}. \tag{10}$$

2.2. Experiments. Method a)

Experiments were performed on materials: austenitic steel 316L, aluminum alloy Al99,5, electrolytic copper and polystyrene cut out from various sheet thickness. Dimensions of the specimen were chosen in such a way, so that the influence of its edges on the surface temperature distribution was negligible. In order to ensure high and homogeneous emissivity the specimen surface was coated by graphite paint. The emissivity of graphite is *0.86.* The surface of the specimen was uniformly heated using the halogen lamp of the pulse energy of *6 kJ.* Pulse duration was *3 ms* and the lamp to specimen distance was equal to *0.5m*. Temperature distribution on the opposite surface vs. time was measured by the Titanium 560M infrared thermographic system (Cedip Company) with InSb detector. The spectral range of the detector was (*3.6-5.1*) *μm*. The thermal sensitivity of the system at *25 °C* is *20 mK*. The thermal images (640 x 512

http://dx.doi.org/10.21611/qirt.2014.093

pixels) were recorded with the frequency *100 Hz*. The IR camera and stimulating lamp were located at the rear side of the specimen (figure 2).

The thermal image is a surface distribution of infrared radiation power, emitted by the tested surface. This distribution depends on emissivity of the surface and temperature distribution on this surface. Thus, knowing the emissivity of graphite (*ε=0.86*), the surface temperature distribution was determined and the average value of temperature of the specimen's back surface as a function of time was obtained. This function is presented in the figure 3 In the figure 3, there is a fragment of a graph, in which the average temperature value of the tested surface is constant. Hence, it may be concluded that the process of convection does not develop until *t=0.72s* , and it may be not taken into consideration. The maximum temperature is equal to the surface temperature, which would be reached after a long time if heat convection did not take place, $T_{max} = T_{\infty}$. The dependence of $\ln\left(T_{\max} - T\right)$ on *t* is presented in figure 4.

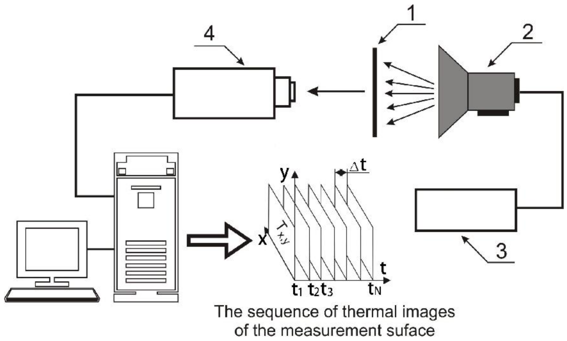

Fig. 2. *The scheme of the measuring systems for determining the diffusivity of solids. 1 is specimen, 2 is flash lamp, 3 is power supply, 4 is IR camera, 5 is computer with appropriate software to enable recording thermal images of the specimen surface as functions of time*

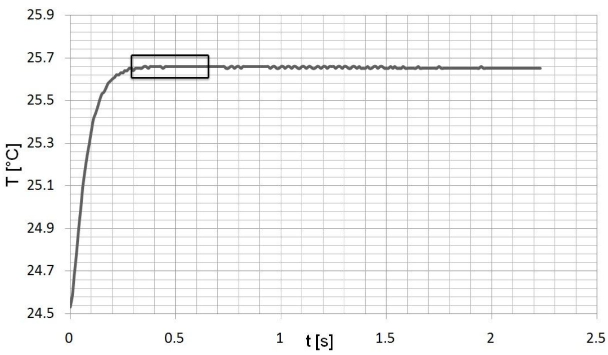

Fig. 3. *Experimentally determined the surface temperature vs. time for austenitic steel 316L. The marked plato shows that the convection development requires a certain time interval*

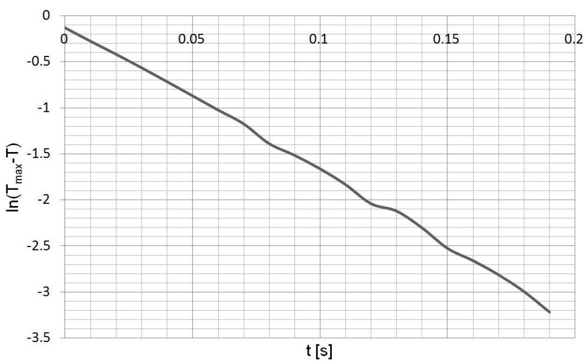

Fig. 4. *The dependence of* $ln\left(T_{max} - T\right)$ *on time for austenitic steel 316L*

The graph of this dependence has been approximated by the straight line:

$$ln(T_{\infty} - T) = -16.13t + 0.078 \tag{11}$$

http://dx.doi.org/10.21611/qirt.2014.093

The comparison of the equation of this line with Eq. (8) (figure 5) shows that

$$\frac{\pi^2 \alpha}{g^2} = 16.13\, s^{-1}. \tag{12}$$

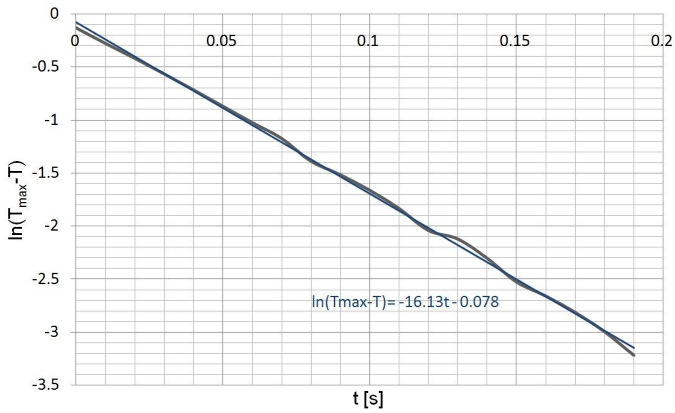

Fig. 5. *The dependence of* $ln\left(T_{max} - T\right)$ *vs. time approximated by the straight line. Austenitic steel 316L*

Thus, after transformation of Eq. (12) and substituting the value of the steel sheet thickness *g=1,5 mm*, the value of thermal diffusivity of 316L steel was determined,

$$\alpha = 16.13 \frac{0.0015^2}{\pi^2} = 3.67 \times 10^{-6} \frac{m^2}{s}. \tag{13}$$

The same procedure was applied for all mentioned materials. The obtained results were compared with literature values [2] of thermal diffusivity. This comparison is presented in table 1.

Table 1. *Comparison of the thermal diffusivity of the tested materials obtained in the presented work with the values for this materials given in the literature [2]*

Material	Thermal diffusivity given in the literature *[m²/s]*	Thermal diffusivity obtained in the presented work *[m²/s]*	Difference *[%]*
Austenitic steel 316L	3.71×10^{-6}	3.68×10^{-6}	0.8
Polystyrene	1.138×10^{-7}	1.172×10^{-7}	2.9
Aluminum alloy Al99,5	9.43×10^{-5}	9.10×10^{-5}	3.5
Electrolytic copper	1.13×10^{-4}	1.18×10^{-4}	4.2

The values of the thermal diffusivity of tested materials determined using method a) are very approximate to literature ones obtained by using more complicated methods. But the method a) has the fallowing drawbacks:

- The heat source function has the form of Dirac deltas in time and in space, whereas the duration of the heat impulse is finite.
- The solutions are usually have a form of series.

In case of above limitations attempt was made to find a solution of the heat conduction equation in a form of analytical function. Therefore method b) is proposed.

2.2. Method b)

In the method, stimulation of the plate surface by heat pulse was taken into account in boundary condition as the temperature dependence of time for *z = 0*:

$$T\left(z = 0, t\right) = \varphi\left(t\right), \tag{14}$$

Mathematical form of the function $\varphi\left(t\right)$ was found on the basis of temperature time evolution of the stimulated surface of the plate, which was measured by means of IR Thermography (figure 6). Thus the formulated condition is entirely consistent with the experiment. The initial condition and the boundary condition for *z=g* are the same as in the method a),

http://dx.doi.org/10.21611/qirt.2014.093

$$\frac{\partial T}{\partial z}\left(z=g,t\right)=0, T\left(z,t=0\right)=T_0. \tag{15}$$

In this case, we solve the homogeneous equation:

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial z^2}. \tag{16}$$

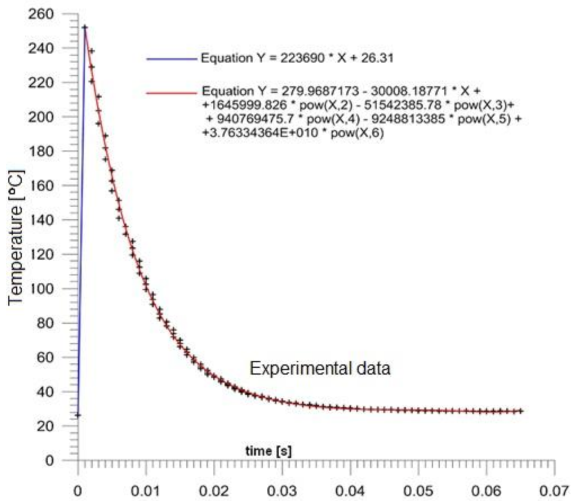

Fig.6. *Function* $\varphi\left(t\right)$ *measured experientially*

To find the solution the so-called heat penetration distance (depth) function $\gamma\left(t\right)$ is introduced, through which the temperature depends on time [3, 4], i.e.: $T\left(z,t\right)=\hat{T}\left(z,\gamma\left(t\right)\right)$ and its properties are such that at time $t<t_1$ the part of plate described by the condition $z>\gamma\left(t\right)$ is at reference temperature T_0, and there is no heat transferred beyond the point $z=\gamma\left(t\right)$; here t_1 denotes the time moment at which the penetration distance function is equal *g*:

$$\gamma\left(t_1\right)=g. \tag{17}$$

Hence the solution will be searched in two steps: 1. for $t<t_1$, and 2. for $t\geq t_1$. Moreover, following the integral heat balance method of von Karman [5], in each of these steps will be look for under two simplifying assumptions:

- distribution of the temperature inside the plate will be assumed as a polynomial of order *i* = 2 and
- the partial differential equation (14) will be substituted by its integral form:

$$\int_{v(t)}^{w(t)}\frac{\partial T\left(z,t\right)}{\partial t}dz=\int_{v(t)}^{w(t)}\alpha\frac{\partial^2 T\left(z,t\right)}{\partial z^2}dz, \text{ with } \left(v(t),w(t)\right)\subset\left[0,g\right]. \tag{18}$$

After integration of RHS of (18) and use of the Leibniz rule of integral differentiation we obtain:

$$\frac{d}{dt}\int_{v(t)}^{w(t)}T\left(z,t\right)dz-T\left(w\left(t\right),t\right)\frac{dw}{dt}+T\left(v\left(t\right),t\right)\frac{dv}{dt}=\alpha\left[\frac{\partial T\left(z,t\right)}{\partial z}\Big|_{z=w(t)}-\frac{\partial T\left(z,t\right)}{\partial z}\Big|_{z=v(t)}\right]. \tag{19}$$

http://dx.doi.org/10.21611/qirt.2014.093

2.2.1 First step $\left(\gamma(t) \le g\right)$

In this step we assume that $v(t)=0$ and $\omega(t)=\gamma(t)$. Then, due to definition of the function $\gamma(t)$ and our initial condition (2), we have: $T\left(w(t),t\right)=0$ and $\frac{\partial T}{\partial z}\Big|_{z=w(t)}=0.$ Moreover, since $\frac{dv}{dt}=0$, and introducing the mean temperature value of the plate part $[0,\gamma]$ by $<T>_1(t):=\frac{1}{\gamma(t)}\int_0^{\gamma(t)} T(z,t),$ for $0\le\gamma\le g,$ (20)

the heat balance equation (19) reduce to $\frac{d}{dt}\left[\gamma(t)\left(<T>_1(t)-T_0\right)\right]=-\alpha\frac{\partial T(z,t)}{\partial z}\Big|_{z=0}.$ (21)

According to simplification the distribution of $T(z,t)$ in this step is assumed in the form of the polynomial

$$T(z,t)=a_0+a_1z+a_2z^2=:\ T_1(z,t) \text{ where } a_i \text{ depend on } t \tag{22}$$

From the initial and boundary conditions (14) and (15) we have constrains: $T(0,t)=\varphi(t),\ T\left(\gamma(t),t\right)=T_0,\ \frac{\partial T_1}{\partial z}\Big|_{z=\gamma}=0$

Substituting above conditions to the expression for $T_1(z,t)$, we obtain:

$$a_0=\varphi(t),\ a_0+a_1\gamma(t)+a_2\left(\gamma(t)\right)^2=T_0,\ a_1+2a_2\gamma(t)=0\ .$$

Solving this system of equations we have: $a_2=-\frac{1}{2\gamma}a_1=-\frac{1}{2\gamma^2}\left(2T_0-2a_0\right).$

Hence $T(z,t)=T_1(z,t)=\varphi(t)+\frac{z}{\gamma}\left(2T_0-2\varphi(t)\right)+\frac{1}{2}\left(\frac{z}{\gamma}\right)^2\left(2T_0-2\varphi(t)\right),$ and after reduction we get up with

$$T(z,t)=T_1(z,t)=T_0+\left(\varphi(t)-T_0\right)\left(1-\frac{z}{\gamma(t)}\right)^2. \tag{23}$$

Now, having the temperature distribution in the plate part $[0,\gamma]$ the mean temperature value $<T>_1$ can be calculated by use of (20). After integration of: $<T>_1(t):=\frac{1}{\gamma(t)}\int_0^{\gamma(t)}T(z,t)\,dz=\frac{1}{\gamma(t)}\int_0^{\gamma(t)}\left[T_0+\left(\varphi(t)-T_0\right)\left(1-\frac{z}{\gamma(t)}\right)^2\right]dz,$

we have:

$$<T>_1(t)=T_0+\frac{1}{3}\left(\varphi(t)-T_0\right). \tag{24}$$

Now we substitute the expression (24) and (23) into (21), to get

$$\frac{d}{dt}\left[\gamma(t)\frac{1}{3}\left(\varphi(t)-T_0\right)\right]=\frac{2\alpha}{\gamma(t)}\left[\varphi(t)-T_0\right].$$

After reduction and introduction new functions: $\Theta(t)=\varphi(t)-T_0$ and $\xi(t):=\left(\gamma(t)\right)^2,$ (25)

we get $\Theta(t)\gamma(t)\frac{d\gamma(t)}{dt}+\left(\gamma(t)\right)^2\frac{d\Theta(t)}{dt}=6\alpha\Theta(t),$ or $\Theta\frac{d\xi}{dt}+2\xi(t)\frac{d\Theta(t)}{dt}=12\alpha\Theta(t),$ (26)

The solution of (26) with the initial condition $\gamma(t)\ \Big|_{t=0}=0$ is

http://dx.doi.org/10.21611/qirt.2014.093

$$\xi(t) = (\gamma(t))^2 = 12\alpha \frac{\int_0^t exp\left(\int_0^s p(\sigma)\,d\sigma\right)ds}{\int_0^t p(s)\,ds}, \tag{27}$$

where $p(s) : \frac{2}{\Theta(s)} \frac{d\Theta(s)}{ds} = 2\frac{d(log\,\Theta(s))}{ds}.$

Integrating (27), we obtain :

$$\xi(t) = (\gamma(t))^2 = 12\alpha \frac{\int_0^t exp\left(log\left(\frac{\Theta(s)}{\Theta(0)}\right)^2\right)ds}{exp\left(log\left(\frac{\Theta(s)}{\Theta(0)}\right)^2\right)} = 12\alpha \frac{\int_0^t \left(\frac{\Theta(s)}{\Theta(0)}\right)^2 ds}{\left(\frac{\Theta(s)}{\Theta(0)}\right)^2}. \tag{28}$$

To get the explicit form of the solution for $\gamma(t)$ we need the form of $\varphi(t)$. In our case this form can be written as

$$\varphi(t) = b_1 t + T_0, \text{ if } 0 \le t \le 0.001s = t_M, \tag{29a}$$

$$\varphi(t) = \sum_{i=0}^{6} c_i (-t)^i \text{ if } t_M \le t \le 0.06s, \tag{29b}$$

with positive constants b_1 i c_i, i = 0, 1, …,6. Hence for $\Theta(s)$ we can write

$$\Theta(s) = b_1 s, \text{ if } 0 \le s \le 0.001s =: s_M, \tag{30a}$$

$$\Theta(s) = \sum_{i=0}^{6} c_i (-s)^i - T_0, \text{ if } s_M \le s \le 0.06s, \tag{30b}$$

The last expressions together with (14) give as for $\xi(t)$ in the first sub-domain $0 \le t \le 0.001s$

$$\xi(t) = 12\alpha \frac{\int_0^t (b_1 s)^2\,ds}{b_1^2 t^2} = 12\alpha \frac{b_1^2 t^3}{3 b_1^2 t^2} = 4\alpha t, \quad \text{and for } \gamma(t)$$

$$\gamma(t) = 2\sqrt{(\alpha t)}. \tag{31}$$

In the second sub-domain $s_M < s \le 0.06$ we get:

$$\gamma(t) = \frac{\sqrt{12\alpha G(t)}}{\Theta(t)} = 2\frac{\sqrt{3\alpha G(t)}}{\Theta(t)}, \tag{32}$$

where $G(t) = \int_0^t (\Theta(s))^2\,ds = \int_0^t \left(\sum_{t=0}^{6} c_i (-s)^2 - T_0\right)^2 ds.$

We end this step with the condition on the value of t_1 which reads

$$\frac{2\sqrt{3\alpha G(t_1)}}{\varphi(t_1) - T_0} = g. \tag{33}$$

http://dx.doi.org/10.21611/qirt.2014.093

2.2.2 Second step $\left(t \geq t_1 \text{ with } \gamma(t_1) = g\right)$

In this step in (18) we assume that $v(t) = 0$ and $w(t) = g$. Introducing the mean temperature value of the whole plate

$[0, g]$ by $<T>_2(t) := \frac{1}{g}\int_0^g T(z,t)\,dz,$ (34)

the heat balance equation (19) reduces to $g\frac{d}{dt} <T>_2(t) = \alpha\left(\frac{\partial T(z,t)}{\partial z}\Big|_{z=g} - \frac{\partial T(z,t)}{\partial z}\Big|_{z=0}\right).$ (35)

Due to the boundary condition (14) and (15) the first term on RHS vanishes and the last equation simplifies to

$$g\frac{d}{dt} <T>_2(t) = -\alpha\frac{\partial T(z,t)}{\partial z}\Big|_{z=0}. \qquad (36)$$

According to our simplification the distribution of $T(x,t)$ in this step is assumed in form of the polynomial

$$T(z,t) = k_0 + k_1 z + k_2 z^2 =: T_2(z,t) \text{ where } k_i \text{ depend on } t. \qquad (37)$$

From the initial and boundary conditions (14) and (15) we have constrains: $T(0,t) = \varphi(t), \frac{\partial T(z,t)}{\partial z}\Big|_{z=g} = 0.$
Let us introduce new unknown function describing temperature values at the rear surface of the plate.

$$u(t) := T(z = g, t), \qquad (38)$$

then for the time dependent coefficients $k_i, i = 0,1,2$ we have the system of equations:

$$k_0 + k_1 g + k_2 g^2 = u(t), \frac{\partial u}{\partial z}\Big|_{z=g} = k_1 + 2k_2 g = 0,\ k_0 = \varphi(t).$$

Solving this system we get $k_1 = -2k_2 g,\ u(t) - \varphi(t) = -k_2 g^2, k_0 = \varphi(t)$.

Hence we have: $T(z,t) = T_2(z,t) = \varphi(t) - \frac{2zg}{g^2}\left(\varphi(t) - u(t)\right) + \frac{z^2}{g^2}\left(\varphi(t) - u(t)\right),$

and after reduction, we and up with $T(z,t) = T_2(z,t) = u(t) + \left(\varphi(t) - u(t)\right)\left(1 - \frac{z}{g}\right)^2.$ (39)

Now, having the temperature distribution in the whole slab we may calculate $\frac{\partial T(z,t)}{\partial z}\Big|_{z=0}$ and the mean temperature value $<T>_2$ by use of (34). For the gradient we obtain:

$$\frac{\partial T(z,t)}{\partial z}\Big|_{z=0} = -\frac{2}{g}\left(\varphi(t) - u(t)\right)\left(1 - \frac{z}{g}\right)\Big|_{z=0} = \frac{2}{g}\left(u(t) - \varphi(t)\right). \qquad (40)$$

After integration of $<T>_2 := \frac{1}{g}\int_0^g T(z,t)\,dz = \frac{1}{g}\int_0^g \left[u(t) + \varphi(t) - u(t)\left(1 - \frac{z}{g}\right)^2\right]dz$, we have:

http://dx.doi.org/10.21611/qirt.2014.093

$$< T >_2 \left(t\right) = u\left(t\right) + \frac{-g}{3g}\left(\varphi\left(t\right) - u\left(t\right)\right)\left(1 - \frac{z}{g}\right)^3 \Big|_0^g = u\left(t\right) + \frac{1}{3}\left(\varphi\left(t\right) - u\left(t\right)\right) = \frac{1}{3}\left(\varphi\left(t\right) + 2u\left(t\right)\right) \quad (41)$$

Now, we substitute the expression (41) into (36), to get $\frac{1}{3}\frac{d\varphi}{dt} - \frac{2\alpha}{g^2}\varphi\left(t\right) + \frac{2}{3}\frac{du}{dt} = -\frac{2\alpha}{g^2}u\left(t\right).$ (42)

Introducing $\mu := \frac{\alpha}{g^2}$, we obtain: $\frac{d\varphi}{dt} - 6\mu\varphi\left(t\right) + 2\frac{du}{dt} = -6\mu u\left(t\right)$, or

$$\frac{du}{dt} + 3\mu u\left(t\right) = 3\mu\varphi\left(t\right) - \frac{1}{2}\frac{d\varphi\left(t\right)}{dt} \quad (43)$$

The equation (43) is valid for $t \geq t_1$, where t_1 is the time at which solution $\gamma\left(t\right)$ of (26) given by (32) is equal to *g* - the thickness of the plate. To solve (43) we need an initial condition at the moment. However, the function $u\left(t\right)$, defined by (38) gives the value of the temperature at the rear face, at the time t_1 is the first moment at which the heat reaches that face and before that time the rear was at the reference temperature. It means that

$$u\left(t_1\right) = T_0 \quad (44)$$

The solution of (43) is: $u\left(t\right) = e^{-3\mu\left(t-t_1\right)}\left[\int_{t_1}^{t}\left(3\mu\varphi\left(s\right) - \frac{1}{2}\frac{d\varphi}{ds}\right)e^{-3\mu\left(s-t_1\right)}ds + T_0\right].$ (45)

To find $u\left(t\right)$ we need the explicit of $3\mu\varphi\left(s\right) - \frac{1}{2}\frac{d\varphi\left(s\right)}{ds}$, knowing that φ is given by (29*a*) and (29*b*). For derivative of φ we have: $\frac{d\varphi\left(s\right)}{ds} = b_1$ if $0 \leq s \leq 0.001s =: s_M$ and $\frac{d\varphi\left(s\right)}{ds} = -\sum_{i=1}^{6} ic_i\left(-s\right)^{i-1}$, if $0 \leq s \leq 0.06s$

Hence we get:

$$3\mu\varphi\left(s\right) - \frac{1}{2}\frac{d\varphi\left(s\right)}{ds} = 3\mu b_1 s + \left(3\mu T_0 - \frac{1}{2}b_1\right), \text{ if } 0 \leq s \leq s_M \quad (46a)$$

and $3\mu\varphi\left(s\right) - \frac{1}{2}\frac{d\varphi\left(s\right)}{ds} = 3\mu c_6 s^6 + \sum_{i=1}^{5}\left(3\mu c_i + \frac{\left(i+1\right)}{2}c_{i+1}\right)\left(-s\right)^i + 3\mu c_0 + \frac{c_1}{2}$, if $0 \leq s \leq 0.06s$ (46b)

Substituting (46a) and (46b) to (45) we obtain:

for $0 \leq s \leq s_M$, $u\left(t\right) = e^{-3\mu\left(t-t_1\right)}\left[\int_{t_1}^{t}\{3\mu\left(b_1 s + T_0\right) - \frac{1}{2}b_1\}e^{-3\mu\left(s-t_1\right)}ds + T_0\right]$ (47a)

and for $s_M < s \leq 0.06\,s$,

$$u\left(t\right) = e^{-3\mu\left(t-t_1\right)}\left[\int_{t_1}^{t}\{3\mu c_6 s^6 + \sum_{i=1}^{5}\left(3\mu c_i + \frac{\left(i+1\right)}{2}c_{i+1}\right)\left(-s\right)^i + 3\mu c_0 + \frac{c_1}{2}\}e^{-3\mu\left(s-t_1\right)}ds + T_0\right]. \quad (47b)$$

Expressions in RHS of (47a) and (47b) can be written as the sum of integrals:

http://dx.doi.org/10.21611/qirt.2014.093

for $0 \le s \le s_M$, $u(t) = e^{-3\mu(t-t_1)} \left[3\mu b_1 \int_{t_1}^{t} s e^{-3\mu(s-t_1)} ds + 3\mu T_0 t \int_{t_1}^{t} e^{-3\mu(s-t_1)} ds - \frac{1}{2} b_1 \int_{t_1}^{t} e^{-3\mu(s-t_1)} ds + T_0 \right]$ (48a)

and for $s_M < s \le 0.06\, s$,

$$u(t) = e^{-3\mu(t-t_1)} \left[3\mu c_6 \int_{t_1}^{t} s^6 e^{-3\mu(s-t_1)} ds + \sum_{i=1}^{5} \left(3\mu c_i + \frac{(i+1)}{2} c_{i+1} \right) \int_{t_1}^{t} (-s)^i e^{-3\mu(s-t_1)} ds \right] +$$
$$+ e^{-3\mu(t-t_1)} \left[\left(3\mu c_0 + \frac{c_1}{2} \right) \int_{t_1}^{t} e^{-3\mu(s-t_1)} ds + T_0 \right]. \tag{48b}$$

Having the explicit form of the integrals in (48a) and (48b) we may pass to their calculation. To end this we first notice that we are still delaying with the polynomial of order 6 in the temporal independent variable. Hence we need the general form of the indefinite integral

$$\int s^i \, exp(\eta s) \, ds$$

with *i* = 0, 1, …..6 and constant $\eta = -3\mu$. It is not difficult to prove, using the mathematical induction principle, that

$$\int s^i e^{\eta t} dt = \frac{1}{\eta} \left[s^i - \frac{i}{\eta} s^{i-1} + \frac{i(i-1)}{\eta^2} s^{i-2} - \frac{i(i-1)(i-2)}{\eta^3} s^{i-3} + ... + (-1)^{i-1} \frac{i!}{\eta^{i-1}} s + (-1)^i \frac{i!}{\eta^i} \right] e^{\eta s}. \tag{49}$$

Using (49) to calculate the integrals in the expressions (48a) and (48b) we get the form of the function *u*(*t*). According to (38), this function describes time evolution of temperature at the rear surface of the tested plate. Then substituting $\mu = \frac{\alpha}{g^2}$ into $u(t)$ and choosing α so that the theoretically found function $u(t)$ coincides with the experimentally measured temperature dependence on time, for the rear surface, the thermal diffusivity of the tested material can be determined. Such procedure will be the subject of a further work.

3. Conclusions

It has been showed that inverse problem in thermal diffusivity determination using pulsed IRT can be solved by comparison measured time evolution of the temperature of the opposite specimens' surface to the stimulated one with theoretical solution of the heat conduction equation at initial and boundary conditions consisted with the experiment. Two methods of that solution has been presented. The first one although its limitations gives quite satisfactory results. Second one still require further development.

REFERENCES

[1] Kochanowski K., Oliferuk W., Płochocki Z., Adamowicz A., „Determination of thermal diffusivity of austenitic steel using pulsed infrared thermography." Archives of Metallurgy and Materials. 2014, (3) 59.

[2] Goodfellow, "Metals, Alloys, Compounds, Ceramics, Polymers, Composites" Catalogue 1993/94

[3] Arpaci V. S., "Conduction Heat Transfer " Heat and Mass Transfer" Advances in Heat Transfer" , Addison – Wesley, 1966..

[4] Biot M. A., C., "New methods in heat flow analysis with application to flight structures" , J. Aeronaut . Sci., vol 24, pp. 857-873, 1957.

[5] Karman T. „ Über laminare und turbulente Reibung", Zeitschrift für Angewandte Mathematik und Mechanik, vol.1, 233-252, 1921.